Brief Summary of NOWS Theoretical Backgrounds

Power spectral density matrix of longitudinal wind velocity fluctuation

- Power spectral density function of longitudinal wind velocity fluctuations – two-sided form
  - Kaimal et al. (1972); Simiu (1974); Simiu and Scanlan (1996),

\[
S_{rr}(z, \omega) = \frac{1}{2} \frac{200}{2\pi} u_*^2 \frac{z}{U(z)} \frac{1}{1 + 50 \frac{\omega \sigma}{2\pi U(z)}}^{5/3}
\]

where, \( z \) = height, \( \omega \) = circular frequency (rad/s); \( u_* \) = friction velocity; \( U(z) \) = mean wind speed at height \( z \)

Coherence function (two-dimensional)

- Davenport (1967); Simiu and Scanlan (1996)

\[
f_{rs}(\omega) = \exp \left[ -\frac{\omega}{2\pi} \frac{C_z \Delta z}{\frac{1}{2} [U(z_r) + U(z_s)]} \right] \cdot \exp \left[ -\frac{\omega}{2\pi} \frac{C_z \Delta x}{\frac{1}{2} [U(x_r) + U(x_s)]} \right]
\]

where, \( x, z \) = horizontal and vertical directions, respectively; \( \Delta z = |z_r - z_s|; \Delta x = |x_r - x_s|; C_z \),

\( C_z \) = a constant, generally taken 10 and 16 for structural design viewpoint, respectively.

Cross-spectral density function

- Co-spectrum (quadratic term of wind is ignored)

\[
S_{rs}(\omega) = \sqrt{S_{rr}(\omega) \cdot S_{ss}(\omega)} \exp(-f_{rs}(\omega))
\]

Power spectral density matrix \( S(\omega) \) : two-dimensional, \( n \)-variate

\[
S(\omega) = \begin{bmatrix}
S_{11}(\omega) & S_{12}(\omega) & \cdots & S_{1n}(\omega) \\
S_{21}(\omega) & S_{22}(\omega) & \cdots & S_{2n}(\omega) \\
\vdots & \vdots & \ddots & \vdots \\
S_{n1}(\omega) & S_{n2}(\omega) & \cdots & S_{nn}(\omega)
\end{bmatrix}
\]
Simulation schemes of wind velocity fluctuations

1. Discrete frequency function with FFT
   - Wittig and Sinha (1975)

Discrete time series can be simulated using the following model:

\[
y_p(n\Delta t) = \frac{1}{N} \sum_{k=0}^{N} Y_p(k\Delta f) \exp \left( j \frac{2\pi kn}{N} \right)
\]

\[
Y_p(k\Delta f) = \sum_{j=0}^{p} H_{pj}(k\Delta f) \xi_{ik} \sqrt{2f_c N}
\]

where,

- \( H_{pj}(k\Delta f) \): a lower triangular matrix by Cholesky decomposition of one-sided power spectral density function \( S(f) \);
- \( \xi_{ik} = \xi_{ik} + j\eta_{ik} \) = complex Gaussian random number with zero mean and 0.5 variance;
- \( \Delta t = \frac{1}{2f_c} \); \( f_c \) = Nyquist frequency
2. Schur decomposition approach with AR (autoregressive)

– Di Paola (1998); Di Paola and Gullo (2001)

The $n$-variate stochastic vector process $V(t)$ can be decomposed into a summation of $n$-variate fully coherent normal vectors $Y_j(t)$ independent of each other:

$$V(t) = \sum_{j=1}^{n} Y_j(t)$$

Let $\Psi(\omega)$ be the eigenmatrix of $S(\omega)$ whose columns are the eigenvectors (real and orthogonal), then, following relationship holds:

$$\Psi^T(\omega)S(\omega)\Psi(\omega) = \Lambda(\omega)$$

$$\Psi^T(\omega)\Psi(\omega) = I$$

Vectors $Y_j(t)$ can be described as:

$$Y_j(t) = \int_{-\infty}^{\omega} S(\omega)e^{i\omega t} dB_j(\omega) = \int_{-\infty}^{\omega} \psi_j(\omega)\sqrt{\Lambda(\omega)}e^{i\omega t} dB_j(\omega)$$

Let define the frequency domain $[\omega_0, \omega_c]$, where $\omega_0$ and $\omega_c$ are lower and upper cut-off frequencies, and subdivided the domain into $M$ parts $\omega_0 = \Omega_0, \Omega_1, \ldots, \Omega_m = \omega_c$

With third-order polynomial approximation of eigenvectors $\psi_j^{(s)}(\omega)$,

$$\psi_j^{(s)}(\omega) = N_j^{(s)}l(\omega), \quad \Omega_{s-1} \leq \omega \leq \Omega_s$$

where, $l(\omega) = [1 \quad \omega \quad \omega^2 \quad \omega^3]$.

Accordingly, vectors $Y_j(t)$ can be expressed as:

$$Y_j(t) = \sum_{s=1}^{M} N_j^{(s)} \int_{\Omega_{s-1}}^{\Omega_s} l(\omega)\sqrt{\Lambda(\omega)}e^{i\omega t} dB_j^{(s)}(\omega) = \sum_{s=1}^{M} N_j^{(s)} U_j^{(s)}(t)$$

where, $U_j^{(s)}(t) = \sum_{s=1}^{M} \int_{\Omega_{s-1}}^{\Omega_s} l(\omega)\sqrt{\Lambda(\omega)}e^{i\omega t} dB_j^{(s)}(\omega)$

Using the standard generation via AR(autoregressive) model:

$$U_j^{(s)}(t_k) = \sum_{i=1}^{p} a_{j,i}^{(s)} U_j^{(s)}(t_{k-i}) + \sigma_j^{(s)} W_j^{(s)}(t_k)$$

where $r = 1,\ldots,4$; $a_{j,i}^{(s)}$ = parameters of the AR model; $\sigma_j^{(s)}$ = variances of the input; $W_j^{(s)}$ = normal random variables with zero mean and unit variance; $p$ = AR model order

AR parameters can be evaluated by using Yule-walker scheme, that is, autocorrelation method.

It is worth noting that in this study, $M = 1$ and $p = 4$ are used to generate wind velocity fluctuations.
3. Ergodic spectral representation method
– Deodatis (1996); Ding et al. (2006)

Power spectral density matrix $S(\omega)$ can be decomposed into the following product:

$$S(\omega) = H(\omega)H^T(\omega)$$

where, $H(\omega)$ is a lower triangular matrix by Cholesky decomposition of $S(\omega)$

Stochastic process $V_j(t)$ can be described by using following trigonometric series:

$$V_j(t) = 2\sum_{m=1}^{N}\sum_{l=1}^{M} H_{jm}(\omega)\sqrt{\Delta\omega}\cos[\omega_{ml} t - \theta_{jm}(\omega_{ml}) + \Phi_{ml}]$$

To take advantage of the FFT technique, above equation can be rewritten as follows:

$$V_j(p\Delta t) = \text{Re}\left\{\sum_{m=1}^{N}h_{jm}(p\Delta t)\exp\left[i\left(\frac{m\Delta\omega}{n}\right)(p\Delta t)\right]\right\}$$

where, $N = \text{number of n-variate simulation}; j = 1,2,\ldots,N; p = 0,1,\ldots,n\times(M-1); M = 2N;$

$$h_{jm}(p\Delta t) = \begin{cases} g_{jm}(p\Delta t) & \text{for } p = M, M+1,\ldots,2M-1 \\ g_{jm}[(p-M)\Delta t] & \text{for } p = M, M+1,\ldots,2M-1 \\ \vdots & \vdots \\ g_{jm}[(p-nM)\Delta t] & \text{for } p = nM, n(M+1),\ldots,nM-1 \end{cases}$$

where,

$$g_{jm}(p\Delta t) = \sum_{l=0}^{M-1}B_{jm l}\exp\left[ilp\frac{2\pi}{M}\right] \quad p = 0,1,\ldots,M-1$$

$$B_{jm l} = 2[H_{jm}(\omega_{ml})]\sqrt{\Delta\omega}\exp[-i\theta_{jm}(\omega_{ml})]\exp(i\Phi_{ml})$$

$$\omega_{ml} = (l-1)\Delta\omega + \frac{m}{n}\Delta\omega \quad m = 1,2,\ldots,n; \quad l = 1,2,\ldots,N$$

$$\theta_{jm}(\omega_{ml}) = \tan^{-1}\left\{\frac{\text{Im}(H(\omega_{ml}))}{\text{Re}(H(\omega_{ml}))}\right\}$$

$$\Phi_{ml} = \text{independent random phase angles distributed uniformly over the interval } [0, 2\pi]$$

Note: $g_{jm}(p\Delta t)$ can be obtained from inverse FFT of $B_{jm l}$
4. Conventional spectral representation method

Shinozuka and Deodatis (1991)

Power spectral density matrix $S(\omega)$ can be decomposed into the following product:

$$S(\omega) = H(\omega)H^T(\omega)$$

where, $H(\omega)$ is a lower triangular matrix by Cholesky decomposition of $S(\omega)$.

Stochastic process $V(t)$ can be described by using following trigonometric series:

$$V(t) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos \left( \omega_n t + \phi_n \right)$$

To take advantage of the FFT technique, above equation can be rewritten as follows:

$$V(p\Delta t) = \text{Re} \left\{ \sum_{n=0}^{M-1} B_n \exp \left[ i (n\Delta\omega)(p\Delta t) \right] \right\}$$

where, $A_n = \sqrt{(2 \cdot \Delta \omega)} \cdot |H(\omega)|$, $N$ = number of n-variate simulation; $p = 0, 1, \ldots, M - 1$;

$M = 2N$; $B_n = 2 |H(\omega)| \cdot \exp \left( i \phi_n \right)$; $\Delta\omega = \omega_c / N$; $\omega_c$ = upper cut-off frequency [rad/sec]; $\phi_n$ = independent random phase angles distributed uniformly over the interval [0, $2\pi$]
Selected references


